Four Fundamental Subspaces

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The $m \times n$ System: $Ax = b$

**Axiom.** The system $Ax = b$ is solvable if and only if the vector $b$ can be expressed as the linear combination of the columns of $A$ (lies in $\text{Span}[\text{columns of } A]$ or geometrically lies in the subspace defined by columns of $A$).
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**Definition.** The set of non-trivial solutions $x \neq \theta$ to the homogeneous system $Ax = \theta$ is itself a vector space called the null space of $A$, denoted by $\mathcal{N}(A)$. 
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**Remark.** *All the possible cases in the solution of the simple scalar equation* \( \alpha x = \beta \) *are below:*

- \([\alpha \neq 0]: \forall \beta \in \mathbb{R}, \exists x = \frac{\beta}{\alpha} \in \mathbb{R}\) (nonsingular case),
- \([\alpha = \beta = 0]: \forall x \in \mathbb{R}\) are the solutions (undetermined case),
Remark. All the possible cases in the solution of the simple scalar equation $\alpha x = \beta$ are below:

- $[\alpha \neq 0:] \forall \beta \in \mathbb{R}, \exists x = \frac{\beta}{\alpha} \in \mathbb{R}$ (nonsingular case),
- $[\alpha = \beta = 0:] \forall x \in \mathbb{R}$ are the solutions (undetermined case),
- $[\alpha = 0, \beta \neq 0:]$ there is no solution (inconsistent case).
Let us consider a possible $LU$ decomposition of a given $A \in \mathbb{R}^{m \times n}$ with the help of the following example:

\[
A = \begin{bmatrix}
1 & 3 & 3 & 2 \\
2 & 6 & 9 & 5 \\
-1 & -3 & 3 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 3 & 3 & 2 \\
0 & 0 & 3 & 1 \\
0 & 0 & 6 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 3 & 3 & 2 \\
0 & 0 & 3 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} = U.
\]

The final form of $U$ is upper-trapezoidal.
The $m \times n$ System: $Ax = b$

**Definition.** An upper-triangular (lower-triangular) rectangular matrix $U$ is called **upper-(lower-)trapezoidal** if all the nonzero entries $u_{ij}$ lie on and above (below) the main diagonal, $i \leq j$ ($i \geq j$). An upper-trapezoidal matrices has the following “echelon” form:

$$
\begin{bmatrix}
\circ & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & \circ & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \circ & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \circ & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$
The $m \times n$ System: $Ax = b$

In order to obtain such an $U$, we may need row interchanges, which would introduce a permutation matrix $P$. Thus, we have the following theorem.
The \( m \times n \) System: \( Ax = b \)

**Theorem.** For any \( A \in \mathbb{R}^{m \times n} \), there is a permutation matrix \( P \), a lower-triangular matrix \( L \), and an upper-trapezoidal matrix \( U \) such that \( PA = LU \).
The $m \times n$ System: $Ax = b$

**Theorem.** For any $A \in \mathbb{R}^{m \times n}$, there is a permutation matrix $P$, a lower-triangular matrix $L$, and an upper-trapezoidal matrix $U$ such that $PA = LU$.

**Definition.** In any system $Ax = b \iff Ux = c$, we can partition the unknowns $x_i$ as basic (dependent) variables those that correspond to a column with a nonzero pivot $\circ$, and free (nonbasic, independent) variables corresponding to columns without pivots.
The \( m \times n \) System: \( Ax = b \)

We can state all the possible cases for \( Ax = b \) as we did in the previous remark without any proof.
Theorem. Suppose the $m$ by $n$ matrix $A$ is reduced by elementary row operations and row exchanges to a matrix $U$ in echelon form. Let there be $r$ nonzero pivots; the last $m - r$ rows of $U$ are zero. Then, there will be $r$ basic variables and $n - r$ free variables as independent parameters. The null space, $\mathcal{N}(A)$, composed of the solutions to $Ax = \theta$, has $n - r$ free variables.
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Theorem. Suppose the $m$ by $n$ matrix $A$ is reduced by elementary row operations and row exchanges to a matrix $U$ in echelon form. Let there be $r$ nonzero pivots; the last $m - r$ rows of $U$ are zero. Then, there will be $r$ basic variables and $n - r$ free variables as independent parameters. The null space, $\mathcal{N}(A)$, composed of the solutions to $Ax = \theta$, has $n - r$ free variables. Solutions exist for every $b$ if and only if $r = m$ ($U$ has no zero rows), and $Ux = c$ can be solved by back-substitution.
Theorem. Suppose the $m$ by $n$ matrix $A$ is reduced by elementary row operations and row exchanges to a matrix $U$ in echelon form. Let there be $r$ nonzero pivots; the last $m - r$ rows of $U$ are zero. Then, there will be $r$ basic variables and $n - r$ free variables as independent parameters. The null space, $\mathcal{N}(A)$, composed of the solutions to $Ax = \theta$, has $n - r$ free variables.

If $r < m$, $U$ will have $m - r$ zero rows. If one particular solution $\hat{x}$ to the first $r$ equations of $Ux = c$ (hence to $Ax = b$) exists, then $\hat{x} + \alpha \hat{x}, \forall \hat{x} \in \mathcal{N}(A) \setminus \{\theta\}, \forall \alpha \in \mathbb{R}$ is also a solution.
The \( m \times n \) System: \( Ax = b \)

**Theorem.** Suppose the \( m \) by \( n \) matrix \( A \) is reduced by elementary row operations and row exchanges to a matrix \( U \) in echelon form. Let there be \( r \) nonzero pivots; the last \( m - r \) rows of \( U \) are zero. Then, there will be \( r \) basic variables and \( n - r \) free variables as independent parameters. The null space, \( \mathcal{N}(A) \), composed of the solutions to \( Ax = \theta \), has \( n - r \) free variables.

**Definition.** The number \( r \) is called the **rank** of \( A \).
The $m \times n$ System: $Ax = b$

In order to obtain such an $U$, we may need row interchanges, which would introduce a permutation matrix $P$. Thus, we have the following theorem.
Theorem. For any $A \in \mathbb{R}^{m \times n}$, there is a permutation matrix $P$, a lower–triangular matrix $L$, and an upper-trapezoidal matrix $U$ such that $PA = LU$. 
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Remark. If we rearrange the columns of $A$ so that all basic columns containing pivots are listed first, we will have the following partition of $U$: 
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**Remark.** If we rearrange the columns of $A$ so that all basic columns containing pivots are listed first, we will have the following partition of $U$:

$$A = [B|N] \rightarrow U = \begin{bmatrix} U_B & U_N \\ O \end{bmatrix} \rightarrow V = \begin{bmatrix} I_r & V_N \\ O \end{bmatrix}$$

where $B \in \mathbb{R}^{m \times r}$, $N \in \mathbb{R}^{m \times (n-r)}$, $U_B \in \mathbb{R}^{r \times r}$, $U_N \in \mathbb{R}^{r \times (n-r)}$, $O$ is an $(m - r) \times n$ matrix of zeros, $V_N \in \mathbb{R}^{r \times (n-r)}$, and $I_r$ is the identity matrix of order $r$. $U_B$ is upper-triangular, thus non-singular.
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**Remark.** If we rearrange the columns of $A$ so that all basic columns containing pivots are listed first, we will have the following partition of $U$:

$$A = [B|N] \rightarrow U = \begin{bmatrix} U_B & U_N \\ O & \end{bmatrix} \rightarrow V = \begin{bmatrix} I_r & V_N \\ O & \end{bmatrix}$$

If we continue from $U$ and use elementary row operations to obtain $I_r$ in the $U_B$ part, like in the Gauss-Jordan method, we will arrive at the reduced row echelon form $V$. 
**Definition.** The *row space* of $A$ is the space spanned by rows of $A$. It is denoted by $\mathcal{R}(A^T)$.

$$\mathcal{R}(A^T) = \text{Span}(\{a_i\}_{i=1}^m) = \{ y \in \mathbb{R}^m : y = \sum_{i=1}^m \alpha_i a_i \}$$

$$= \{ d \in \mathbb{R}^m : \exists y \in \mathbb{R}^m \ni y^T A = d^T \}.$$
The row space of $A$

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**Proposition.** The row space of $A$ has the same dimension $r$ as the row space of $U$ and the row space of $V$. They have the same basis, and thus, all the row spaces are the same.
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**Proposition.** The row space of $A$ has the same dimension $r$ as the row space of $U$ and the row space of $V$. They have the same basis, and thus, all the row spaces are the same.

**Proof.** Each elementary row operation leaves the row space unchanged. \qed
**Definition.** The *column space* of $A$ is the space spanned by the columns of $A$. It is denoted by $\mathcal{R}(A)$.

$$\mathcal{R}(A) = \text{Span} \{a^j\}_{j=1}^n = \left\{ y \in \mathbb{R}^n : y = \sum_{j=1}^n \beta_j a^j \right\}$$

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**Proposition.** The dimension of column space of $A$ equals the rank $r$, which is also equal to the dimension of the row space of $A$. The number of independent columns equals the number of independent rows. A basis for $\mathcal{R}(A)$ is formed by the columns of $B$. 
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**Definition.** The *rank* is the dimension of the row space or the column space.
The null space (kernel) of $A$

**Proposition.**

\[
\mathcal{N}(A) = \{ x \in \mathbb{R}^n : Ax = \theta (Ux = \theta, Vx = \theta) \} \\
= \mathcal{N}(U) = \mathcal{N}(V).
\]
The null space (kernel) of $A$

**Proposition.** The dimension of $N(A)$ is $n - r$, and a base for $N(A)$ is the columns of $T = \begin{bmatrix} -V_N \\ I_{n-r} \end{bmatrix}$.

Proof.
The null space (kernel) of $A$

**Proposition.** The dimension of $\mathcal{N}(A)$ is $n - r$, and a base for $\mathcal{N}(A)$ is the columns of $T = \begin{bmatrix} -V_N \\ I_{n-r} \end{bmatrix}$.

**Proof.**

$$Ax = \theta \iff Ux = \theta \iff Vx = \theta \iff x_B + V_Nx_N = \theta.$$ 

The columns of $T = \begin{bmatrix} -V_N \\ I_{n-r} \end{bmatrix}$ is linearly independent because of the last $(n - r)$ coefficients.
The null space (kernel) of \( A \)

**Proposition.** The dimension of \( \mathcal{N}(A) \) is \( n - r \), and a base for \( \mathcal{N}(A) \) is the columns of \( T = \begin{bmatrix} -V_N \\ I_{n-r} \end{bmatrix} \).

**Proof.** Is their span \( \mathcal{N}(A) \)?

Let \( y = \sum_j \alpha_j T^j \), \( Ay = \sum_j \alpha_j (-V_N^j + V_N^j) = \theta \). Thus, \( \text{Span}(\{T^j\}_{j=1}^{n-r}) \subseteq \mathcal{N}(A) \). \( \square \)
The null space (kernel) of $A$

**Proposition.** The dimension of $\mathcal{N}(A)$ is $n - r$, and a base for $\mathcal{N}(A)$ is the columns of $T = \begin{bmatrix} -V_N \\ I_{n-r} \end{bmatrix}$.

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Let $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} \in \mathcal{N}(A)$. Then,
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**Proof.** Is $\text{Span}(\{T^j\}_{j=1}^{n-r}) \supseteq \mathcal{N}(A)$?

\[
Ax = \theta \iff x_B + V_N x_N = \theta \iff \\
x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} -V_N \\ I_{n-r} \end{bmatrix} x_N \in \text{Span}(\{T^j\}_{j=1}^{n-r})
\]
**Proposition.** The dimension of \( \mathcal{N}(A) \) is \( n - r \), and a base for \( \mathcal{N}(A) \) is the columns of \( T = \begin{bmatrix} -V_N \\ I_{n-r} \end{bmatrix} \).

**Proof.** Is \( \text{Span}(\{T^j\}_{j=1}^{n-r}) \supseteq \mathcal{N}(A) \)? Thus, \( \text{Span}(\{T^j\}_{j=1}^{n-r}) \supseteq \mathcal{N}(A) \).
The left null space of $A$

**Definition.** The subspace of $\mathbb{R}^m$ that consists of those vectors $y$ such that $y^T A = \theta$ is known as the **left null space** of $A$. $\mathcal{N}(A^T) = \{ y \in \mathbb{R}^m : y^T A = \theta \}$. 
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**Proposition.** The left null space $\mathcal{N}(A^T)$ is of dimension $m - r$, where the basis vectors are the last $m - r$ rows of $L^{-1}P$ of $PA = LU$ or $L^{-1}PA = U$. 

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**Proof.** $\bar{A} = [A|I_m] \rightarrow \bar{V} = \left[ \begin{array}{c|c} I_r & V_N \\ \hline O & L^{-1}P \end{array} \right]$ Then,

$$(L^{-1}P) = \left[ \begin{array}{c} S_I \\ S_{II} \end{array} \right],$$

where $S_{II}$ is the last $m - r$ rows of $L^{-1}P$. 
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**Definition.** The subspace of $\mathbb{R}^m$ that consists of those vectors $y$ such that $y^T A = \theta$ is known as the **left null space** of $A$. $\mathcal{N}(A^T) = \{ y \in \mathbb{R}^m : y^T A = \theta \}$.

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**Proof.**

$\bar{A} = [A|I_m] \rightarrow \bar{V} = \begin{bmatrix} I_r & V_N \\ O & L^{-1}P \end{bmatrix}$

Then,

$S_{II}A = \theta$.  

\[ \qed \]
Four Fundamental Subspaces

- $R(A) \oplus N(A) = \mathbb{R}^n$
- $R(A^T) \oplus N(A^T) = \mathbb{R}^m$

$A$ is an $m \times n$ matrix with rank $r$. The subspaces $R(A)$ and $N(A)$ are complementary, as are $R(A^T)$ and $N(A^T)$.
Fund. Theorem of Linear Algebra

**Theorem.** \( \mathcal{R}(A^T) = \) row space of \( A \) with dimension \( r \);
\[ \mathcal{N}(A) = \text{null space of } A \text{ with dimension } n - r; \]
\[ \mathcal{R}(A) = \text{column space of } A \text{ with dimension } r; \]
\[ \mathcal{N}(A^T) = \text{left null space of } A \text{ with dimension } m - r; \]
Collaborative Work

**Definition.** Let $GF(2)$ be the field with $+$ and $\times$ (addition and multiplication modulo 2 on $\mathbb{Z}_2$)

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Collaborative Work
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Consider the node–edge incident matrix of the given graph

\[ G = (V, E) \text{ over } GF(2), \quad A \in \mathbb{R}^{\|V\| \times \|E\|}: \]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
a & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
f & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
g & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
h & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
i & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]
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The addition $+$ operator helps to point out the end points of the path formed by the added edges. For instance, if we add the first and ninth columns of $A$, we will have
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Collaborative Work

The addition $+$ operator helps to point out the end points of the path formed by the added edges. For instance, if we add the first and ninth columns of $A$, we will have $[1, 0, 0, 1, 0, 0, 0, 0, 0]^T$, which indicates the end points (nodes $a$ and $d$) of the path formed by edges one and nine.
Collaborative Work

Find the reduced row echelon form of $A$ working over $GF(2)$.

$$[A|I_9] = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
a & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
c & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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Collaborative Work

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\end{bmatrix}
\]

\(a + b \rightarrow b;\ a + b \rightarrow a;\ b + c \rightarrow c;\ c + d \rightarrow d;\)
\(d + e \rightarrow e;\ e + f \rightarrow f;\ f + g \rightarrow g;\ g + i \rightarrow i;\ h + i \rightarrow i\)
Collaborative Work

\[ a + b \rightarrow b; \ a + b \rightarrow a; \ b + c \rightarrow c; \ c + d \rightarrow d; \]
\[ d + e \rightarrow e; \ e + f \rightarrow f; \ f + g \rightarrow g; \ g + i \rightarrow i; \ h + i \rightarrow i \]

\[ [A||I_9] \rightarrow \begin{bmatrix} I_8 & N & C \\ 0 & D \end{bmatrix} = \]

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Four Fundamental Subspaces – p.21/25
Collaborative Work

Each basis corresponds to a spanning tree $T$ in $G = (V, E)$, where $T \subset E$ connects every vertex and $\|T\| = \|V\| - 1$. Here, we have $T = \{1, 2, 3, 4, 5, 6, 7, 8\}$. 
Collaborative Work

Each row represents a fundamental cocycle (cut) in the graph.
Collaborative Work

Each row represents a fundamental cocycle (cut) in the graph. In the tree, we term one node as the root (node i), and we can associate an edge of the tree with every node like $1 \rightarrow b$, $2 \rightarrow a$, $3 \rightarrow c$, $4 \rightarrow d$, $\cdots$, $8 \rightarrow h$ as if we hanged the tree to the wall by its root.
Collaborative Work

Each row represents a fundamental cocycle (cut) in the graph. Then, if the associated edge (say edge 6) in the tree for the node (say \( f \)) is removed, we partition the nodes into two sets as

\[ V_1 = \{a, b, c, d, e, f\} \] and \[ V_2 = \{g, h, i\} . \]
Collaborative Work

Each row represents a fundamental cocycle (cut) in the graph. The nonzero entries in $N_f$ correspond to edges 10, 12, 13, defining the set of edges connecting nodes in different parts of this partition or the cut. The set of such edges are termed as fundamental cocycle.
Collaborative Work

Each column represents a fundamental cycle. If we add the edge identified by $I_5$ part into $T$, we will create a cycle defined by the nonzero elements of $y^j$. 
Collaborative Work

Each column represents a fundamental cycle. If we add the edge identified by $I_5$ part into $T$, we will create a cycle defined by the nonzero elements of $y^j$. Let us take edge 10:
Collaborative Work

The first 8 columns of $A$ form a **basis** for column space $\mathcal{R}(A)$.

$$[A||I_9] \rightarrow \begin{bmatrix} I_8 & N & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i \end{bmatrix}$$

$$\begin{array}{cccccccccccccccccccccccccc}
\hline
& a & b & c & d & e & f & g & h & i \\
\hline
a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
f & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
g & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

Four Fundamental Subspaces – p.25/25
Collaborative Work

The first 8 columns of $A$ form a **basis** for column space $\mathcal{R}(A)$.

$$[A||I_9] = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
a & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
c & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
f & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
g & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
h & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
i & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}$$
Collaborative Work

The columns of matrix $Y = \begin{bmatrix} -N \\ I_5 \end{bmatrix}$ is a basis for the null space $\mathcal{N}(A)$. 

$$[A||I_9] \rightarrow \begin{bmatrix} I_8 & N & C \\ 0 & D \end{bmatrix} =$$

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Collaborative Work

The rows of $C$ constitute a basis for the row space $\mathcal{R}(A^T)$.

$$[A||I_9] \rightarrow \begin{bmatrix} I_8 & N & C \\ 0 & D \end{bmatrix} =$$

\[
\begin{array}{cccccccccccccccc}
\text{a} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{b} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{c} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{d} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{e} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{f} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{g} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Finally, the row(s) of matrix $D$ is (are) the basis vectors for the left-null space $\mathcal{N}(A^T)$.

$$\begin{bmatrix} A | I_9 \end{bmatrix} \rightarrow \begin{bmatrix} I_8 & N & C \\ 0 & D \end{bmatrix} =$$

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$